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Reflection and refraction of electromagnetic waves by the plane boundary of a non-linear medium: application of the simple-wave theory

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Abstract. A unified treatment of electromagnetic non-linear wave propagation in terms of simple waves in an isotropic, non-dispersive and non-linear medium is presented. Based on this theory the problem of reflection and refraction of a plane wave incident on a semi-infinite non-linear medium is investigated systematically and solved rigorously. As a special case, the reflection and transmission of constant-amplitude waves is studied in detail. Two different methods are presented for the numerical solution of the resulting systems of non-linear reflection and propagation equations. A number of numerical results are presented.

1. Introduction

Recent theoretical and experimental investigations in the field of laser physics and non-linear transmission lines have increased the interest in the analysis of non-linear electromagnetic wave problems. In laser physics we mention the harmonic generation, optical rectification, optical mixing and parametric amplification in passive, non-linear media (Bloembergen 1965), and the self-focusing of optical beams (Askar'yan 1962, Chiao *et al* 1964), where the permittivity is assumed to be proportional to the time-averaged value of the square of the electric field strength. Research in the field of non-linear transmission lines involves the investigation of analogues to wave propagation in ferroelectric ceramics (see Auld *et al* 1962, Scott 1970) and the propagation of pulses in a neuristor (Reible and Scott 1975).

One of the tools to measure the non-linear properties of a dielectric is to observe the reflected wave that results from an incident pulse. The simplest geometry in this regard is the plane boundary of a specimen of the material. This motivates the study of the non-linear reflection problem by a plane boundary. For certain classes of non-linear materials the pulse distortion upon reflection in the range of pulse widths that are used in practice, is mainly determined by the deviation from linearity, while the dispersion of the material is of minor importance.

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Mathematically the non-linear reflection problem in the presence of dispersion is in general almost intractable, except for perturbation methods applicable to small non-linearities. However, in cases where dispersion can be neglected, exact solutions of the non-linear wave equation can be constructed in the form of '*simple waves*' that can have arbitrary large non-linearities. Therefore the application of the simple-wave theory to the non-linear reflection problem can yield very useful results, the more so since arbitrary large signal amplitudes can be taken into account.

The simple waves are generalisations of travelling waves of constant shape in linear media and can be introduced with the aid of the theory of characteristics. A simple wave can be viewed as a wave travelling with a local velocity that depends on the local field value. Since different field values travel with different velocities, the wave changes in shape as it propagates and eventually shock-wave formation takes place. Simple-wave solutions of a certain class of hyperbolic systems of partial differential equations have been introduced by Courant and Friedrichs (1948) and Jeffrey and Tanuti (1964). Salinger (1923) has introduced the concept of simple waves in the study of non-linear transmission lines. The propagation of shock waves along non-linear transmission lines has been investigated by Landauer (1960a, b) and Katayev (1963). Much other work on this topic has been reported in a book by Scott (1970). Broer (1963, 1964, 1965) has introduced the concept of simple waves in electromagnetics and has used it to obtain an exact solution of the reflection problem in non-linear optics. Simple waves of a special type, the so called '*constant-amplitude*' waves, have been studied by Carroll (1967, 1972) and by Pettini (1969), in particular with reference to their possible use for distortionless transmission of information through a non-dispersive, isotropic, non-linear dielectric.

In the present paper, a unified treatment of electromagnetic non-linear wave propagation in terms of simple waves and a discussion of their properties is presented. The theory deals with non-dispersive, isotropic, non-linear media. A simple wave in such a medium is found to depend on space and time through the variable $u = t - \boldsymbol{s} \cdot \boldsymbol{r}$ only, in which $\boldsymbol{s}$ is the *slowness* vector that does not explicitly depend on either t or $\boldsymbol{r}$. A special study is made of the Poynting vector, in the expression of which extra terms occur that are absent for plane waves in linear media. The extra terms vanish for constant-amplitude waves. Based on our simple-wave theory, a unified treatment of the reflection and refraction problem in the presence of a semi-infinite non-linear medium is presented. In the analysis, non-linear differential reflection and transmission factors are introduced. As a special case, the reflection and transmission of constant-amplitude waves by a semi-infinite non-linear medium is studied. This unified approach to the problem can be considered as a generalisation of previous work by Broer and Carroll.

As a novel element, two different methods are presented for the numerical solution of the resulting systems of non-linear reflection and propagation equations. The first method uses an appropriate iteration scheme. In the second, a piecewise linear approximation to the non-linear constitutive relations is employed to solve the non-linear reflection equations numerically. The latter method considerably reduces the computing time. Various numerical results will be presented: (i) the influence of the degree of non-linearity on the propagation of simple waves evolving from different initial pulse shapes; (ii) a Fourier analysis of the transmitted simple waves for various non-linear profiles when a plane wave varying sinusoidally in time is normally incident on the half-space; (iii) the differential reflection and transmission factors for various angles of incidence and various non-linear profiles.

2. Formulation of the reflection problem

In this section we consider the problem of reflection and transmission of a plane electromagnetic wave propagating in an isotropic, linear medium and incident on the plane boundary separating the linear medium from an isotropic, non-linear medium. The more general problem of reflection of a non-linear wave at the plane interface between two non-linear media is very complicated and can only be solved using the method of characteristics (see Jeffrey and Tanuti 1964). The incident plane wave propagates obliquely with respect to the plane interface of the two media. This plane interface occupies the entire x, y plane at $z=0$, as illustrated in figure 1. The

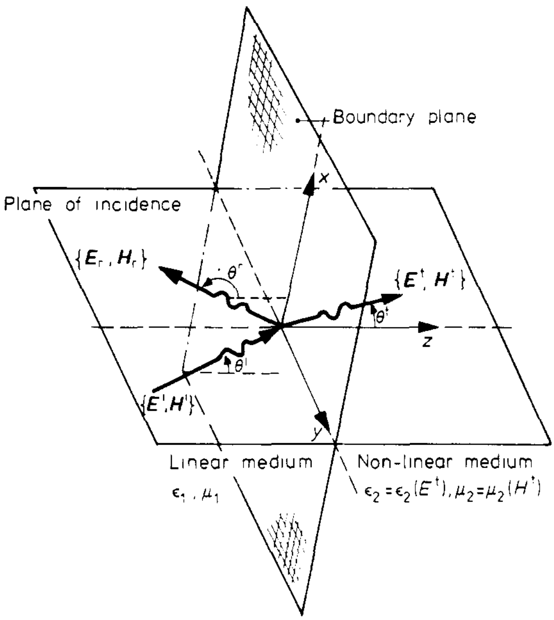

Figure 1. A plane wave incident at an angle θ^i on the plane boundary of a semi-infinite non-linear medium.

non-linear medium occupies the half-space $0<z<\infty$. In the half-space $-\infty<z<0$ a linear, isotropic medium is present. The electromagnetic field quantities $\boldsymbol{E}=\boldsymbol{E}(\boldsymbol{r}, t)$, $\boldsymbol{H}=\boldsymbol{H}(\boldsymbol{r}, t)$, $\boldsymbol{D}=\boldsymbol{D}(\boldsymbol{r}, t)$ and $\boldsymbol{B}=\boldsymbol{B}(\boldsymbol{r}, t)$ satisfy in both domains the time-dependent, source-free electromagnetic field equations

$$\nabla \times \boldsymbol{H} - \partial_t \boldsymbol{D} = \mathbf{0}, \tag{2.1}$$

$$\nabla \times \boldsymbol{E} + \partial_t \boldsymbol{B} = \mathbf{0}, \tag{2.2}$$

$$\nabla \cdot \boldsymbol{D} = 0, \tag{2.3}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{2.4}$$

The electromagnetic properties of the non-dispersive, homogeneous, isotropic, non-linear medium in the domain $0<z<\infty$ are characterized by the constitutive relations $\boldsymbol{D}=\boldsymbol{D}(\boldsymbol{E})$ and $\boldsymbol{B}=\boldsymbol{B}(\boldsymbol{H})$ which are independent of the orientation of the coordinate axes and are therefore described by $D=D(E)$ and $B=B(H)$ with $D=(\boldsymbol{D}\cdot\boldsymbol{D})^{1/2}$,

$E=(\boldsymbol{E}\cdot\boldsymbol{E})^{1/2}$, $B=(\boldsymbol{B}\cdot\boldsymbol{B})^{1/2}$ and $H=(\boldsymbol{H}\cdot\boldsymbol{H})^{1/2}$. This allows the introduction of the *nonlinear differential permittivity* $\epsilon_2=\epsilon_2(E)$ and the *non-linear differential permeability* $\mu_2=\mu_2(H)$ by

$$\epsilon_2(E)=\partial_E D, \qquad \mu_2(H)=\partial_H B, \tag{2.5}$$

respectively. Since these relations hold for each of the components of the constitutive relations $\boldsymbol{D}=\boldsymbol{D}(\boldsymbol{E})$ and $\boldsymbol{B}=\boldsymbol{B}(\boldsymbol{H})$, these can be written as

$$\boldsymbol{D}=\int_0^{\boldsymbol{E}}\epsilon_2(E)\,\mathrm{d}E, \qquad \boldsymbol{B}=\int_0^{\boldsymbol{H}}\mu_2(H)\,\mathrm{d}H. \tag{2.6}$$

The electromagnetic properties of the non-dispersive, homogeneous, isotropic, linear medium in the domain $-\infty<z<0$ are characterized by the relations

$$\boldsymbol{D}=\epsilon_1\boldsymbol{E}, \qquad \boldsymbol{B}=\mu_1\boldsymbol{H}. \tag{2.7}$$

The incident wave is a uniform plane wave in the half-space $-\infty<z<0$. The reflected wave is assumed to be a plane wave in the half-space $-\infty<z<0$. The transmitted wave in the half-space $0<z<\infty$ is assumed to be a simple wave satisfying the equations (A.9) and (A.10). The three waves are of the form $\{\boldsymbol{E},\boldsymbol{H}\}(\boldsymbol{r},t)=\{\boldsymbol{E},\boldsymbol{H}\}(u)$ with $u=t-\boldsymbol{s}\cdot\boldsymbol{r}$. Superscripts i, r, t will refer to the incident, reflected and transmitted waves, respectively. The incident and reflected waves are written as

$$\begin{aligned}\boldsymbol{E}^{\mathrm{i,r}}&=\boldsymbol{H}^{\mathrm{i,r}}\times\boldsymbol{Z}^{\mathrm{i,r}},\\ \boldsymbol{H}^{\mathrm{i,r}}&=\boldsymbol{Y}^{\mathrm{i,r}}\times\boldsymbol{E}^{\mathrm{i,r}}.\end{aligned} \qquad -\infty<z<0. \tag{2.8}$$

The transmitted wave is represented as

$$\begin{aligned}\boldsymbol{E}^{\mathrm{t}}&=\int_0^u\partial_w\boldsymbol{H}^{\mathrm{t}}(w)\times\boldsymbol{Z}^{\mathrm{t}}\{E^{\mathrm{t}}(w),H^{\mathrm{t}}(w)\}\,\mathrm{d}w,\\ \boldsymbol{H}^{\mathrm{t}}&=\int_0^u\boldsymbol{Y}^{\mathrm{t}}\{E^{\mathrm{t}}(w),H^{\mathrm{t}}(w)\}\times\partial_w\boldsymbol{E}^{\mathrm{t}}(w)\,\mathrm{d}w,\end{aligned} \qquad 0<z<\infty, \tag{2.9}$$

in which $E^{\mathrm{t}}=(\boldsymbol{E}^{\mathrm{t}}\cdot\boldsymbol{E}^{\mathrm{t}})^{1/2}$ and $H^{\mathrm{t}}=(\boldsymbol{H}^{\mathrm{t}}\cdot\boldsymbol{H}^{\mathrm{t}})^{1/2}$ and $u=t-\boldsymbol{s}^{\mathrm{t}}\cdot\boldsymbol{r}$. In (2.8) and (2.9) we have

$$\boldsymbol{s}^{\mathrm{i,r}}\cdot\boldsymbol{s}^{\mathrm{i,r}}=\epsilon_1\mu_1, \qquad \boldsymbol{s}^{\mathrm{t}}\cdot\boldsymbol{s}^{\mathrm{t}}=\epsilon_2(E^{\mathrm{t}})\mu_2(H^{\mathrm{t}}), \tag{2.10}$$

while the vectorial wave impedances and wave admittances are introduced through

$$\boldsymbol{Z}=\boldsymbol{s}/\epsilon, \qquad \boldsymbol{Y}=\boldsymbol{s}/\mu, \tag{2.11}$$

see also (A.7). In the non-linear half-space ($0<z<\infty$) the simple wave will travel with a local velocity $v=v(E^{\mathrm{t}},H^{\mathrm{t}})=(\boldsymbol{s}^{\mathrm{t}}\cdot\boldsymbol{s}^{\mathrm{t}})^{-1/2}=\{\epsilon_2(E^{\mathrm{t}})\mu_2(H^{\mathrm{t}})\}^{-1/2}$, the amplitude and direction of which depend on the local field values. While the waveshapes of the incident and reflected waves remain unchanged, the shape of the transmitted wave does change in the non-linear half-space, since in a simple wave different field values travel with different velocities. The relations for the transmitted wave derived thus far cease to hold when shock-wave formation occurs. As a consequence the region of existence of the simple wave in the non-linear half-space is bounded. We next decompose the slowness vectors $\boldsymbol{s}^{\mathrm{i}}$, $\boldsymbol{s}^{\mathrm{r}}$ and $\boldsymbol{s}^{\mathrm{t}}$ into their components parallel to the interface ($\boldsymbol{s}_{\mathrm{T}}^{\mathrm{i}}$, $\boldsymbol{s}_{\mathrm{T}}^{\mathrm{r}}$ and $\boldsymbol{s}_{\mathrm{T}}^{\mathrm{t}}$) and their components perpendicular to the interface (i.e. the

z components), i.e.

$$\boldsymbol{s}^{\mathrm{i,r,t}} = \boldsymbol{s}_{\mathrm{T}}^{\mathrm{i,r,t}} + \boldsymbol{i}_z s_z^{\mathrm{i,r,t}}. \tag{2.12}$$

The boundary conditions at $z = 0$ can only be satisfied identically in x and y if

$$\boldsymbol{s}_{\mathrm{T}}^{\mathrm{i}} = \boldsymbol{s}_{\mathrm{T}}^{\mathrm{r}} = \boldsymbol{s}_{\mathrm{T}}^{\mathrm{t}} = \boldsymbol{s}_{\mathrm{T}}. \tag{2.13}$$

Hence the directions of propagation of the incident, reflected and transmitted waves at $z = 0$ are coplanar. From (2.13) we observe that $\boldsymbol{s}_{\mathrm{T}}^{\mathrm{t}}$ will not depend on E^{t} and H^{t}. As a consequence, the dependence of $\boldsymbol{s}^{\mathrm{t}}$ on E^{t} and H^{t} will only manifest itself in s_z^{t}. This z component of $\boldsymbol{s}^{\mathrm{t}}$ is found from (2.10) and (2.12) as

$$s_z^{\mathrm{t}} = \{\epsilon_2(E^{\mathrm{t}})\mu_2(H^{\mathrm{t}}) - \boldsymbol{s}_{\mathrm{T}}^{\mathrm{t}} \cdot \boldsymbol{s}_{\mathrm{T}}^{\mathrm{t}}\}^{1/2}. \tag{2.14}$$

Application of the boundary conditions at the interface $z = 0$ results, after some manipulations, in a system of four vectorial equations for the unknown vectors $\boldsymbol{E}^{\mathrm{r}}$, $\boldsymbol{H}^{\mathrm{r}}$ and $\boldsymbol{E}^{\mathrm{t}}$, $\boldsymbol{H}^{\mathrm{t}}$, in which $\boldsymbol{E}^{\mathrm{i}}$ and $\boldsymbol{H}^{\mathrm{i}}$ are the known quantities. This system can in principle be solved, although under general conditions the result is rather cumbersome. However, in the following two cases the system of equations will become more tractable.

(i) The electric field of the incident wave is linearly polarised and parallel to the boundary (E-polarisation). Following the same reasoning as in the reflection problem for linear media (see e.g. Stratton 1941), it is assumed that the electric field of the reflected and transmitted waves are also linearly polarised and perpendicular to the plane of incidence. This assumption does not lead to contradictions. From (2.9) it then follows that the magnetic field of the transmitted wave is parallel to the plane of incidence, however not necessarily linearly polarised.

(ii) The magnetic field of the incident wave is linearly polarised and parallel to the boundary (H-polarisation). Similar remarks as made for E-polarisation will apply here, too.

In the two cases we are dealing with independent solutions. It is noticed, however, that the problem of reflection of an arbitrarily polarised incident field cannot be solved by a linear superposition of an E-polarised and an H-polarised field. In both cases we introduce the angle $\theta^{\mathrm{i,r,t}}$ as the angle between the slowness vector $\boldsymbol{s}^{\mathrm{i,r,t}}$ and the positive z axis respectively. The relations between the slowness vectors $\boldsymbol{s}^{\mathrm{i}}$, $\boldsymbol{s}^{\mathrm{r}}$ and $\boldsymbol{s}^{\mathrm{t}}$ and the angles θ^{i}, θ^{r} and θ^{t} respectively are illustrated in figure 2. We further see that $s_z^{\mathrm{r}} = -s_z^{\mathrm{i}}$, or $\theta^{\mathrm{r}} = \pi - \theta^{\mathrm{i}}$. From (2.13) it follows that $\boldsymbol{s}_{\mathrm{T}}^{\mathrm{t}} \cdot \boldsymbol{s}_{\mathrm{T}}^{\mathrm{t}} = \epsilon_1\mu_1 \sin^2(\theta^{\mathrm{i}})$.

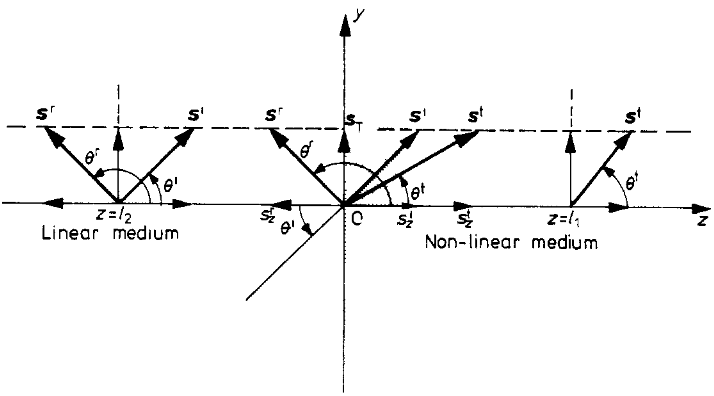

Figure 2. Plane of incidence with the slowness vectors $\boldsymbol{s}^{\mathrm{i}}$, $\boldsymbol{s}^{\mathrm{r}}$ and $\boldsymbol{s}^{\mathrm{t}}$.

2.1. *E*-polarised fields

The boundary conditions at the interface $z=0$ lead to the equations

$$E_x^{\rm i}+E_x^{\rm r}=E_x^{\rm t} \qquad \text{at } z=0, \tag{2.15}$$

$$Y_z^{\rm i}E_x^{\rm i}+Y_z^{\rm r}E_x^{\rm r}=\int_0^u Y_z^{\rm t}\{E^{\rm t}(w),H^{\rm t}(w)\}\,\partial_w E_x^{\rm t}(w)\,{\rm d}w \qquad \text{at } z=0. \tag{2.16}$$

Since $s_{\rm T}^{\rm t}$ will not depend on $E^{\rm t}$ and $H^{\rm t}$ and consequently $Y_y=Y_y(H^{\rm t})$. We obtain from (2.9)

$$H_y^{\rm t}=\int_0^u Y_z^{\rm t}\{E^{\rm t}(w),H^{\rm t}(w)\}\,\partial_w E_x^{\rm t}(w)\,{\rm d}w, \tag{2.17}$$

$$H_z^{\rm t}=-\int_0^u Y_y^{\rm t}\{H^{\rm t}(w)\}\,\partial_w E_x^{\rm t}(w)\,{\rm d}w. \tag{2.18}$$

The equations (2.15)–(2.18) constitute a system of four coupled equations for the unknown quantities $E_x^{\rm r}$, $E_x^{\rm t}$, $H_y^{\rm t}$ and $H_z^{\rm t}$. In (2.16), $H^{\rm t}=[(H_y^{\rm t})^2+(H_z^{\rm t})^2]^{1/2}$ can be expressed in terms of $E_x^{\rm t}$ by solving the implicit equations (2.17) and (2.18). Then using (2.15), (2.11) and (2.16) can be rewritten as

$$Y_1\cos(\theta^{\rm i})(2E_x^{\rm i}-E_x^{\rm i})=\int_0^{E_x^{\rm t}} Y_z^{\rm t}\{E,H^{\rm t}(E)\}\,{\rm d}E \qquad z=0. \tag{2.19}$$

This equation is a simple integral equation from which the transmitted field $E_x^{\rm t}$ at the interface can be computed. In the special case of a non-linear dielectric with $\mu_2(H^{\rm t})=$ constant we have in (2.19), $Y_z^{\rm t}=Y_z^{\rm t}(E)$; in the special case of a non-linear magnetic medium with $\epsilon_2(E^{\rm t})=$ constant we have in (2.19), $Y_z^{\rm t}=Y_z^{\rm t}(H^{\rm t}(E))$.

From (2.15) and (2.16) the *non-linear differential reflection factor* can be defined by

$$\rho^E(E^{\rm t},H^{\rm t})=\frac{{\rm d}E_x^{\rm r}}{{\rm d}E_x^{\rm i}}=\frac{Y_z^{\rm i}-Y_z^{\rm t}(E^{\rm t},H^{\rm t})}{-Y_z^{\rm r}+Y_z^{\rm t}(E^{\rm t},H^{\rm t})} \qquad \text{at } z=0, \tag{2.20}$$

and the *non-linear differential transmission factor* as

$$\tau^E(E^{\rm t},H^{\rm t})=\frac{{\rm d}E_x^{\rm t}}{{\rm d}E_x^{\rm i}}=\frac{Y_z^{\rm i}-Y_z^{\rm r}}{-Y_z^{\rm r}+Y_z^{\rm t}(E^{\rm t},H^{\rm t})} \qquad \text{at } z=0. \tag{2.21}$$

From (2.20) and (2.21) we have $-\rho^E+\tau^E=1$. The expressions found in (2.20) and (2.21) are similar in form to the corresponding Fresnel reflection and transmission factors of an E-polarised plane wave incident on a linear half-space. The angle of incidence at which $\rho^E=\rho^E(E^{\rm t},H^{\rm t})$ vanishes is the Brewster angle; it can be found from (2.20) as

$$\theta_{\rm B}^E=\theta_{\rm B}^E(E^{\rm t},H^{\rm t})=\tan^{-1}\left(\frac{\epsilon_2(E^{\rm t})/\epsilon_1-\mu_2(H^{\rm t})/\mu_1}{\mu_1/\mu_2(H^{\rm t})-\epsilon_2(E^{\rm t})/\epsilon_1}\right)^{1/2}, \tag{2.22}$$

provided that the right-hand side of (2.22) is real.

2.2. *H*-polarised fields

The expressions for H-polarised fields can be obtained from the corresponding results found for E-polarised fields in (2.15)–(2.22) by interchanging the quantities according to

$$\boldsymbol{E}\to\boldsymbol{H}, \qquad \boldsymbol{H}\to\boldsymbol{E}, \qquad \boldsymbol{Z}\to-\boldsymbol{Y}, \qquad \boldsymbol{Y}\to-\boldsymbol{Z}. \tag{2.23}$$

The Brewster angle in the case of H-polarised fields can be found as

$$\theta_{\mathrm{B}}^{H}=\theta_{\mathrm{B}}^{H}(E^{\mathrm{t}},H^{\mathrm{t}})=\tan^{-1}\left(\frac{\mu_2(H^{\mathrm{t}})/\mu_1-\epsilon_1(E^{\mathrm{t}})/\epsilon_1}{\epsilon_1/\epsilon_2(E^{\mathrm{t}})-\mu_2(H^{\mathrm{t}})/\mu_1}\right)^{1/2}, \tag{2.24}$$

provided that the right-hand side of (2.24) is real. When $\mu_2=\mu_1$ we observe from (2.22) and (2.24) that θ_{B}^{E} does not exist, while

$$\theta_{\mathrm{B}}^{H}=\tan^{-1}(\epsilon_2(E^{\mathrm{t}})/\epsilon_1)^{1/2}. \tag{2.25}$$

3. Generation of constant-amplitude waves in a semi-infinite non-linear half-space

In this section we reconsider the reflection problem investigated in § 2. However, the problem is reversed in the sense that the question now is how to choose the incident wave in order that a constant-amplitude wave in the non-linear medium is generated, then the quantities E^{t} and H^{t} are constants and consequently the slowness vector $\boldsymbol{s}^{\mathrm{t}}$ is also a constant vector. The velocity of propagation $v^{\mathrm{t}}=1/s^{\mathrm{t}}$ and the direction of propagation $\hat{\boldsymbol{s}}^{\mathrm{t}}$ will then depend only on the magnitude of the constant amplitude.

In order to facilitate the analysis we write all the field quantities involved in terms of components perpendicular to the plane of incidence (subscript $\perp$) and parallel to the plane of incidence (subscript $\|$):

$$\{\boldsymbol{E},\boldsymbol{H}\}^{\mathrm{i,r,t}}=\{\boldsymbol{E}_{\perp},\boldsymbol{H}_{\perp}\}^{\mathrm{i,r,t}}\boldsymbol{i}_x+\{E_{\|},H_{\|}\}^{\mathrm{i,r,t}}(\hat{\boldsymbol{s}}^{\mathrm{i,r,t}}\times\boldsymbol{i}_x). \tag{3.1}$$

Subsequently we introduce

$$E_{\perp}^{\mathrm{i,r,t}}=E_{\mathrm{l}}^{\mathrm{i,r,t}}\cos[\phi^{\mathrm{i,r,t}}(t-\boldsymbol{s}^{\mathrm{i,r,t}}\cdot\boldsymbol{r})], \tag{3.2}$$

$$E_{\|}^{\mathrm{i,r,t}}=E_{\mathrm{p}}^{\mathrm{i,r,t}}\sin[\phi^{\mathrm{i,r,t}}(t-\boldsymbol{s}^{\mathrm{i,r,t}}\cdot\boldsymbol{r})]. \tag{3.3}$$

The corresponding magnetic fields immediately follow using (A.15). From (3.2) and (3.3) we notice that the transmitted field will be a constant-amplitude wave only if $E_{\mathrm{p}}^{\mathrm{t}}=E_{\mathrm{l}}^{\mathrm{t}}$. As in § 2, the boundary conditions at the interface can only be satisfied when (2.13) holds, which implies that the directions of propagation of the incident, reflected and transmitted waves at $z=0$ are coplanar. Imposing these boundary conditions, we obtain a system of equations from which, $E_{\mathrm{l}}^{\mathrm{t}}=E_{\mathrm{p}}^{\mathrm{t}}=E^{\mathrm{t}}$ being prescribed, $E_{\mathrm{l}}^{\mathrm{i}}$, $E_{\mathrm{r}}^{\mathrm{i}}$, $E_{\mathrm{l}}^{\mathrm{r}}$ and $E_{\mathrm{r}}^{\mathrm{r}}$ can be solved straightforwardly. In particular we find

$$E_{\mathrm{l}}^{\mathrm{i}}=\frac{Y_z^{\mathrm{i}}+Y_z^{\mathrm{t}}}{2Y_z^{\mathrm{i}}}E^{\mathrm{t}}, \tag{3.4}$$

and

$$\frac{E_{\mathrm{p}}^{\mathrm{i}}}{E_{\mathrm{l}}^{\mathrm{i}}}=\frac{(Y_2/Y_1)Y_z^{\mathrm{i}}+(Y_1/Y_2)Y_z^{\mathrm{t}}}{Y_z^{\mathrm{i}}+Y_z^{\mathrm{t}}}. \tag{3.5}$$

The ratio $E_{\mathrm{p}}^{\mathrm{i}}/E_{\mathrm{l}}^{\mathrm{i}}$ in (3.5) can be interpreted as the eccentricity of the (elliptically) polarised incident field. With $E_{\mathrm{l}}^{\mathrm{t}}=E_{\mathrm{p}}^{\mathrm{t}}=E^{\mathrm{t}}$, the constant amplitude of the transmitted wave being prescribed, the ratio $E_{\mathrm{p}}^{\mathrm{i}}/E_{\mathrm{l}}^{\mathrm{i}}$ in (3.5) indicates how to choose the eccentricity of the incident wave for a given angle of incidence. The amplitude of the incident wave then follows from (3.4). In the case of normal incidence it follows immediately from (3.4) and (3.5) that the incident wave should also be a constant-amplitude wave ($E^{\mathrm{i}}_{\mathrm{p}}/E^{\mathrm{i}}_{\mathrm{l}} = 1$), that is, E^{i} and H^{i} should be constant.

4. Numerical methods

Two different methods have been used for the numerical solution of the resulting systems of non-linear equations. In the first method, an appropriate iteration scheme is chosen for the numerical solution of (2.19). The form of this equation suggests a straightforward iterative scheme of the type $E^{\mathrm{t}}_{x,i+1} = \Phi(E^{\mathrm{t}}_{x,i})$, $i = 0, 1, 2, \ldots$. However, a careful investigation of the convergence properties reveals that (2.19) does not satisfy the appropriate Lipschitz condition for convergence (see Traub 1964). An alternative approach is then to write (2.19) in the form $\Psi(E^{\mathrm{t}}_x) = 0$, and to determine subsequently the roots of the latter equation using, e.g., Muller's method (see Traub 1964). Once E^{t}_x and $H^{\mathrm{t}} = H^{\mathrm{t}}(E^{\mathrm{t}}_x)$ at the interface $z = 0$ have been found, the field quantities of the transmitted simple wave in the non-linear medium are computed by a straightforward iteration scheme based on the propagation equation

$$u = t - \boldsymbol{s}^{\mathrm{t}}\{E^{\mathrm{t}}(u), H^{\mathrm{t}}(u)\} \cdot \boldsymbol{r}, \qquad 0 < z < \infty. \tag{4.1}$$

So for the computation of $E^{\mathrm{t}}_x(l, t)$ at a point $z = l > 0$ in the non-linear medium two iterations have to be performed.

A considerable reduction in computing time can be obtained by applying a piecewise linear approximation to the non-linear constitutive relations $D = D(E)$ and $B = B(H)$. For the sake of simplicity we restrict our discussion of this second method to a non-linear dielectric and normal incidence and write

$$\begin{aligned} \partial_{E^{\mathrm{t}}} D^{\mathrm{t}} &= \epsilon_k = \text{constant} \qquad \text{in } E_{k-1} \leqslant E^{\mathrm{t}} < E_k, \\ \partial_{H^{\mathrm{t}}} B^{\mathrm{t}} &= \mu_2 = \text{constant}, \end{aligned} \tag{4.2}$$

(see figure 3). Introducing the (constant) values of Y^{t}_z and Z^{t}_z in the interval

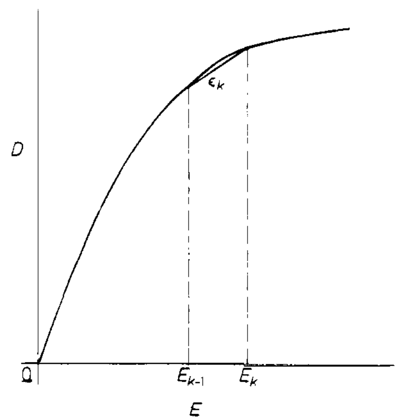

Figure 3. Piecewise linear approximation of the non-linear constitutive relation $D = D(E)$.

(E_{k-1}, E_k), we obtain from (2.19) for $E_x^t(0, t)$ at the interface $z = 0$

$$E_x^t = \frac{2Y_1}{Y_1 + Y_{z,k}^t} E_x^i - \sum_{n=1}^{k-1} \frac{Y_{z,n}^t - Y_{z,k}^t}{Y_1 + Y_{z,k}^t}(E_n - E_{n-1}), \tag{4.3}$$

where k has to be determined such that the inequality $E_{k-1} \leqslant E_x^t < E_k$ holds. The corresponding quantities $H_y^t(0, t)$ and $H_z^t(0, t)$ are found in a similar way:

$$\begin{aligned} H_y^t &= Y_{z,k}^t E_x^t + \sum_{n=1}^{k-1} (Y_{z,n}^t - Y_{z,k}^t)(E_n - E_{n-1}), \\ H_z^t &= -Y_y^t E_x^t. \end{aligned} \tag{4.4}$$

The computational procedure now is as follows. We start with choosing a value for $E_x^t(l, t)$ at a point $z = l$ in the non-linear medium in a certain interval (E_{k-1}, E_k). From (4.2) we then know the value of ϵ_k and compute $v_k = (\epsilon_k \mu_2)^{-1/2}$. Subsequently we compute $\tau_k = l/v_k$, being the time this amplitude constituent of the wave needs to travel from the interface to the point $z = l$. We then calculate $E_x^t(0, t - \tau_k)$ using (4.3) and check whether the computed value lies in the interval (E_{k-1}, E_k). If this is not the case, the procedure is repeated with a modified starting value for $E_x^t(l, t)$. Hence in this second method only a single iteration scheme is needed. In this iteration scheme, a complication may occur. Due to the now discrete character of the propagation velocity in the non-linear medium, different amplitude constituents of the electric field may drift apart in time, and time intervals occur in which the electric field is not defined. This process is illustrated in figure 4. We overcome this difficulty by interpolating between neighbouring values.

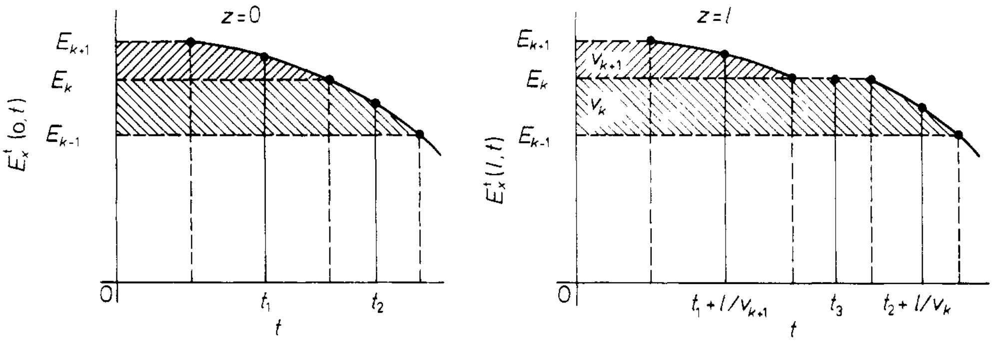

Figure 4. Two values of E_x^t propagating with different velocities v_{k+1} and v_k with respect to each other. At $z = l$ and $t = t_3$, computation of $E_x^t(l, t_3)$ is impossible.

5. Numerical results and discussion

For the configuration given in figure 1, in which the half-space $-\infty < z < 0$ is assumed to be vacuum with ϵ_0 and μ_0 and the half-space $0 < z < \infty$ occupied by a non-linear dielectric with $\epsilon(E)$ and μ_0, several numerical results have been obtained. Using both numerical methods outlined in the previous section, the E-polarised field strength $E_x^t(z, t)$ $(z > 0)$ and $E_x^r(z, t)$ $(z < 0)$ has been calculated for various non-linear profiles and variously shaped pulsed waves normally incident on the boundary $z = 0$. Both methods led to the same results, however considerable reduction in computing time has been obtained applying the method in which the relation between D and E is approximated piecewise linearly. All numerical results presented here have been obtained using the latter method.

In order to illustrate the influence of the non-linearity on the reflection and transmission of an incident plane wave, two different non-linear profiles have been studied in detail, namely

$$\epsilon(E)=\epsilon_0(1+\alpha E^2), \tag{5.1}$$

corresponding to $D(E)=\epsilon_0[E+(\alpha/3)E^3]$ and being a monotonic increasing function with constant curvature; and

$$\epsilon(E)=\epsilon_0[1+\alpha\ \mathrm{sech}^2(E)], \tag{5.2}$$

corresponding to $D(E)=\epsilon_0[E+\alpha\tanh(E)]$ and being a monotonic decreasing function having an inflection point. Both functions are used as approximations to non-linearities in applied physics. For the case of a plane pulsed wave, having the form of the upper half of a sine, normally incident on the boundary $z=0$, the E-polarised transmitted electrical field strength $E_x^t(z,t)$ is plotted in figure 5 at four

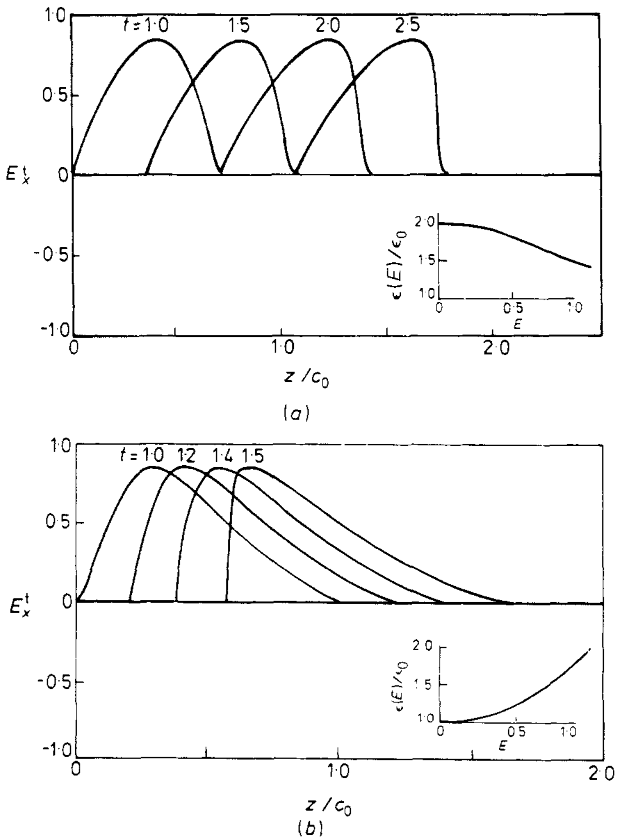

Figure 5. The electric field strength of the transmitted wave $E_x^t(z,t)$ against the distance z/c_0 (expressed in time units) at four different instants t when a pulse, having the form of the upper half of a sine, hits the boundary $z=0$ normally at $t=0$. (a) $\epsilon(E)=\epsilon_0[1+\mathrm{sech}^2(E)]$; ($b$) $\epsilon(E)=\epsilon_0(1+E^2)$.

different instants as a function of the 'normalised' distance z/c_0 (the distance expressed in time units), in which c_0 denotes the velocity of light *in vacuo*. We observe from figure 5(a) that the leading edge of the pulsed wave gradually steepens as the wave penetrates into the non-linear medium. This is in accordance with the fact that for the non-linear profile under consideration the crest of the wave travels faster than the trough. In figure 5(b) the opposite situation occurs. Here the non-linearity is such that the crest of the wave travels slower than the trough and consequently the trailing edge of the pulsed wave steepens as the wave penetrates into the non-linear medium.

Next we have taken the incident field of the form $E^{\mathrm{i}}_x(0, t) = \sin(2\pi f_0 t)$, i.e. simple harmonic with frequency f_0. In figure 6 one time period of the calculated, transmitted field strength $E^{\mathrm{t}}_x(z, t)$ is plotted at three different locations z/c_0 in the non-linear medium. As can be seen from these plots the transmitted field is no longer simple harmonic and consequently generation of higher harmonics must have taken place. This we have examined numerically by calculating the Fourier components $\hat{E}^{\mathrm{t}}_{x,n}(z)$ of the time periodic field strength $E^{\mathrm{t}}_x(z, t)$ with the aid of a fast Fourier transform (FFT). The Fourier series of $E^{\mathrm{t}}_x(z, t)$ can thus be expressed as

$$E^{\mathrm{t}}_x(z, t) = \sum_{n=-\infty}^{\infty} \hat{E}^{\mathrm{t}}_{x,n}(z) \exp(-2\pi \mathrm{i} n f_0 t). \tag{5.3}$$

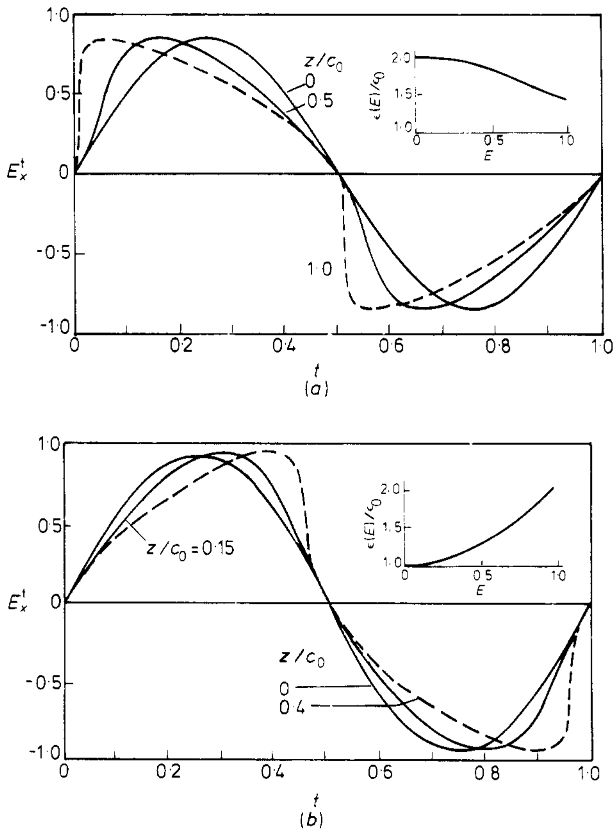

Figure 6. One time period of the electric field strength $E^{\mathrm{t}}_x(z, t)$ at three different locations z/c_0 in the non-linear medium when a sinusoidal field $E^{\mathrm{i}}_x(0, t) = \sin(2\pi f_0 t)$ is normally incident on the boundary $z = 0$. (a) $\epsilon(E) = \epsilon_0[1 + \mathrm{sech}^2(E)]$; ($b$) $\epsilon(E) = \epsilon_0(1 + E^2)$.

The modulus of the complex Fourier components $\hat{E}^{t}_{x,n}(z)$ corresponding to the cases given in figure 6 are listed in table 1.

Table 1. The amplitudes of the harmonics of the complex Fourier components $\hat{E}^{t}_{x,n}(z)$ corresponding to the cases given in figure 6. Incident field: $E^{i}_{x}(0, t) = \sin(2\pi f_0 t)$.

	$\epsilon(E) = \epsilon_0[1 + \mathrm{sech}^2(E)]$			$\epsilon(E) = \epsilon_0(1 + E^2)$		
	$\lvert\hat{E}^{t}_{x,n}(z)\rvert$ at location z/c_0			$\lvert\hat{E}^{t}_{x,n}(z)\rvert$ at location z/c_0		
n \ z/c_0	0	0·5	1·0	0	0·15	0·40
1	0·424	0·419	0·411	0·477	0·475	0·462
3	0·0030	0·061	0·100	0·0072	0·043	0·095
5	0·0001	0·021	0·053	0·0005	0·0095	0·045
7	0·0000	0·0096	0·036	0·0000	0·0031	0·027

First we notice that only odd harmonics are generated. This could be expected since the constitutive relations of the non-linear medium are odd functions. Substitution of the Fourier series (5.3) in $D^{t}_{x} = D^{t}_{x}(E^{t}_{x})$ reveals that the only non-vanishing Fourier components $\hat{D}^{t}_{x,n}(z)$ of the transmitted field generated by the incident field with frequency f_0 are odd numbered. Hence only odd multiples n of the frequency f_0 will occur in the transmitted field and consequently in the reflected field as well. Secondly it is observed that for $n = 3, 5, 7, \ldots$, $\hat{E}^{t}_{x,n}(z)$ increases as we move further into the non-linear medium, while $\hat{E}^{t}_{x,1}(z)$ decreases. Apparently in order to build up the higher harmonics of the field, energy has to be supplied by the fundamental harmonic of the field. For obliquely incident fields we have investigated the dependence of the reflection factors $\rho^{E,H}$ and transmission factors $\tau^{E,H}$ and, if they exist, the Brewster angles $\theta^{E,H}_{B}$ on the magnitude E^{i} of the incident field $E^{i}_{x}(0, t)$. In order to find ρ^{E}, τ^{E} and ρ^{H}, τ^{H} as functions of the magnitude E^{i} of the incident field at $z = 0$, one has to determine the magnitude E^{t} of the transmitted field as a function of E^{i} at $z = 0$ for a given angle of incidence θ^{i}, both for an E-polarised and an H-polarised field. Subsequently E^{t} is substituted in (2.20) and (2.21), yielding the values of ρ^{E}, τ^{E} for E-polarisation and with (2.23) the values of ρ^{H}, τ^{H} for H-polarisation. In figures 7 and 8 the absolute values of the reflection factors $\rho^{E,H}$ and transmission factors $\tau^{E,H}$ are plotted against the magnitude E^{i} of the incident field at $z = 0$ for various angles of incidence θ^{i} and for two non-linear profiles, $\epsilon(E) = \epsilon_0[1 + \alpha\,\mathrm{sech}^2(E)]$ and $\epsilon(E) = \epsilon_0(1 + \alpha E^2)$. In both profiles for the strength parameter α of the non-linearity the values $\alpha = 1$ and $\alpha = 2$ have been taken. The magnitude E^{i} of the incident field has been chosen in the range between 0 and 1 V m^{-1} in order to make the influence of the non-linearity not too pronounced. To give a qualitative view of what happens beyond this range, particularly for the monotonically increasing profile $\epsilon(E) = \epsilon_0(1 + E^2)$, $|\rho^{E,H}|$ and $|\tau^{E,H}|$ are plotted in figure 9 for normal incidence and values of the magnitude of the incident field up to $E^{i} = 5$ V m^{-1}. In order to find the Brewster angles θ^{H}_{B} as a function of E^{i} one has to apply a root-finding procedure to (2.25) since the value of E^{t} is dependent on the angle of incidence θ^{i}. In table 2 the Brewster angles θ^{H}_{B} for an H-polarised field are listed for various values of the magnitude E^{i} of the incident field for the non-linear profiles of $\epsilon(E) = \epsilon_0[1 + \mathrm{sech}^2(E)]$ and $\epsilon(E) = \epsilon_0(1 + E^2)$.

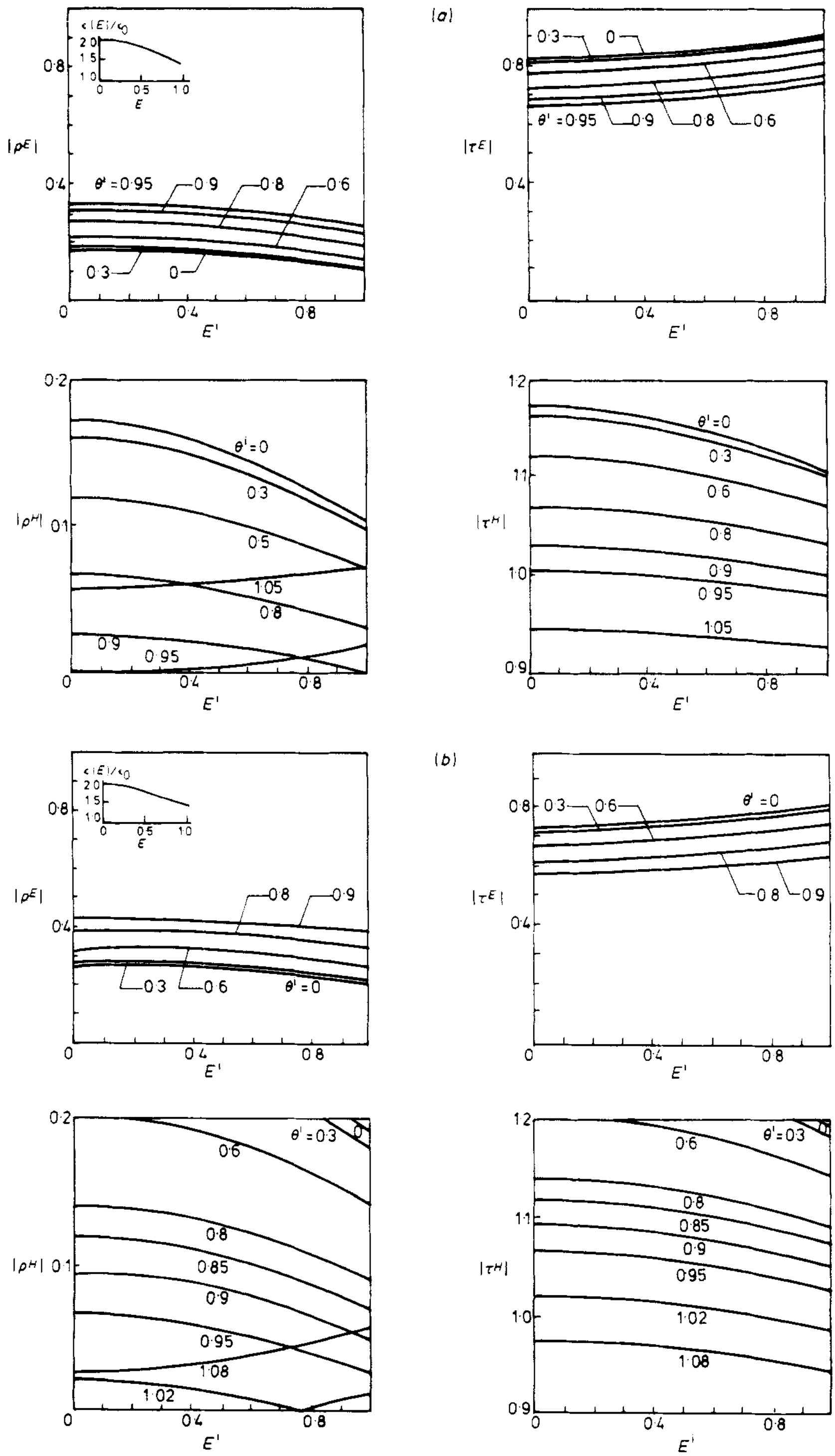

Figure 7. The absolute value of the reflection and transmission factors ρ^E, τ^E and ρ^H, τ^H for E-polarisation and H-polarisation respectively against the amplitude of the incident wave E^i for various angles of incidence θ^i. (a) $|\rho^E|$, $|\tau^E|$ and $|\rho^H|$, $|\tau^H|$ against E^i, $\epsilon(E) = \epsilon_0[1 + \text{sech}^2(E)]$; ($b$) $|\rho^E|$, $|\tau^E|$ and $|\rho^H|$, $|\tau^H|$ against E^i, $\epsilon(E) = \epsilon_0[1 + 2\,\text{sech}^2(E)]$.

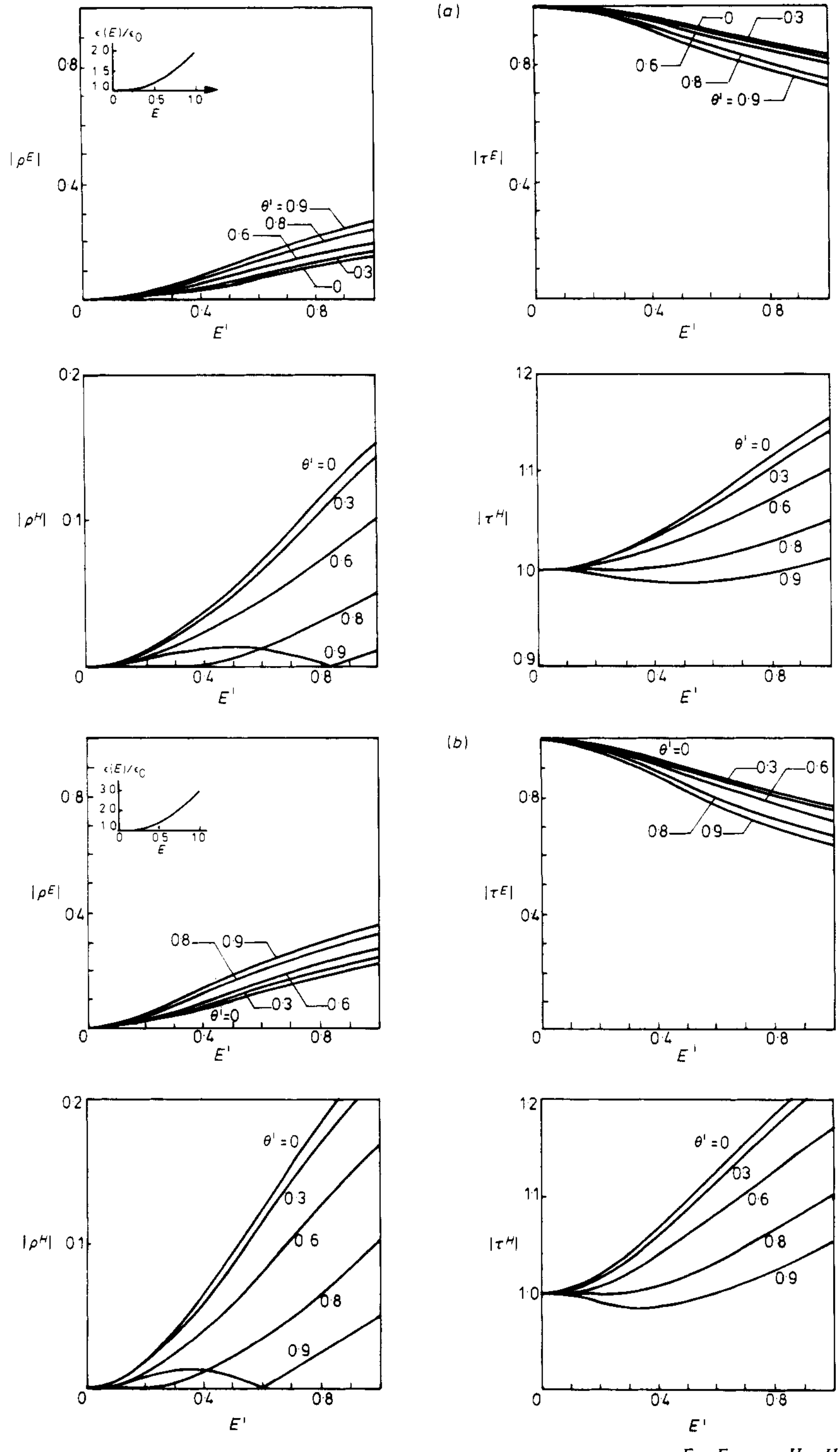

Figure 8. The absolute value of the reflection and transmission factors ρ^E, τ^E and ρ^H, τ^H for E-polarisation and H-polarisation respectively against the amplitude of the incident wave E^i for various angles of incidence θ^i. (*a*) $|\rho^E|$, $|\tau^E|$, and $|\rho^H|$, $|\tau^H|$ against E^i, $\epsilon(E) = \epsilon_0(1 + E^2)$; (*b*) $|\rho^E|$, $|\tau^E|$ and $|\rho^H|$, $|\tau^H|$ against E^i, $\epsilon(E) = \epsilon_0(1 + 2E^2)$.

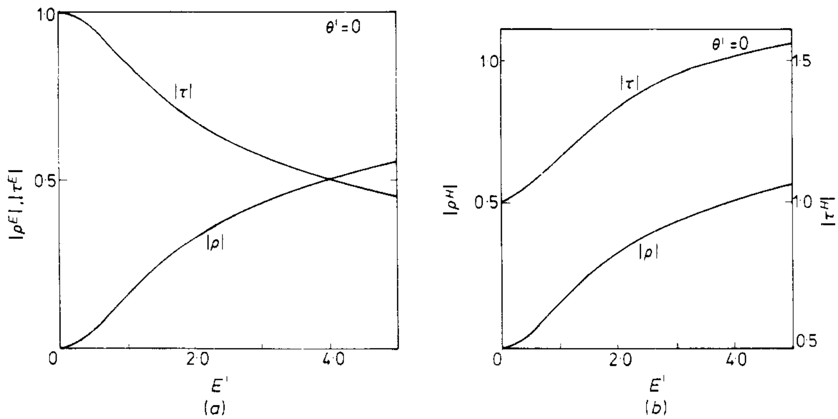

Figure 9. The absolute value of the reflection and transmission factors $\rho^{E,H}$ and $\tau^{E,H}$ against the amplitude E^i of the normally incident field up to $E^i = 5$ V m^{-1}. Non-linear profile: $\epsilon(E) = \epsilon_0(1 + E^2)$. (*a*) $|\rho^E|$, $|\tau^E|$ against E^i; (*b*) $|\rho^H|$, $|\tau^H|$ against E^i.

Table 2. The Brewster angles for various values of the amplitude of the incident wave E^i and two different non-linear profiles.

	$\epsilon(E) = \epsilon_0[1 + \mathrm{sech}^2(E)]$	$\epsilon(E) = \epsilon_0(1 + E^2)$
E^i	θ_B^H (rad)	θ_B^H (rad)
0·2	0·953	0·795
0·4	0·946	0·820
0·6	0·935	0·855
0·8	0·920	0·892
1·0	0·902	0·927
1·2	0·882	0·959
1·5	0·854	1·056

6. Conclusions

In the present paper, a unified treatment of the reflection and refraction problem in the presence of a semi-infinite non-linear medium based on the theory of simple waves is presented. For those classes of non-linear materials where the dispersion can be neglected the method solves the relevant problem for arbitrary large non-linearities and arbitrary large signal amplitudes. In the analysis non-linear differential reflection and transmission factors are introduced. Two different methods are presented for the numerical solution of the resulting systems of non-linear reflection and propagating equations. Most of the numerical results have been obtained applying a piecewise linear approximation to the non-linear constitutive relations, since this method reduces the computing time considerably. With this approach the reflection and transmission of a plane wave obliquely incident on a dispersionless, non-linear half-space with single-valued non-linear constitutive relations can be computed. Unfortunately, this approach cannot be extended to the interesting case of a non-linear slab, since then the field representation in the slab cannot be expressed in terms of simple waves. We have solved the problem of the non-linear slab by means of an integral-equation method in the space–time domain. Results will be reported separately.

Acknowledgments

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Appendix. The theory of simple waves in a non-dispersive non-linear medium

Simple waves in a source-free domain with a non-dispersive, isotropic, non-linear medium are introduced as those special solutions of the field equations (2.1)–(2.4) together with the constitutive equations (2.5) and (2.6) (the subscript '2' in ϵ_2 and μ_2 is omitted in this appendix) for which the field quantities only depend on the space and time coordinates through

$$u = t - \boldsymbol{s} \cdot \boldsymbol{r}, \tag{A.1}$$

in which the *slowness* $s = (\boldsymbol{s} \cdot \boldsymbol{s})^{1/2}$ has to be determined. For a simple wave we can write

$$\{\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}\}(r, t) = \{\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}\}(u). \tag{A.2}$$

In figure 10 the mapping of $(\boldsymbol{r}, t)$ onto u for a given value of $\boldsymbol{s}$ is illustrated. From this illustration it follows that the operators $\boldsymbol{\nabla}$ and ∂_t map as

$$\boldsymbol{\nabla} \to -\boldsymbol{s}\,\partial_u, \qquad \partial_t \to \partial_u. \tag{A.3}$$

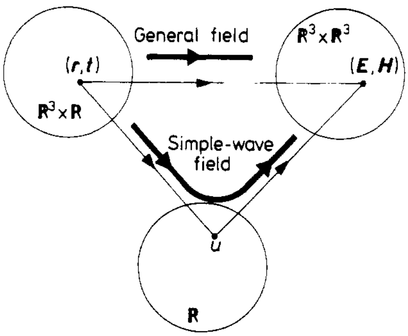

Figure 10. Illustration of the simple-wave field mapping.

With (A.3), the equations (2.1)–(2.6) are written as

$$-\boldsymbol{s} \times \partial_u \boldsymbol{H} - \epsilon(E)\,\partial_u \boldsymbol{E} = \boldsymbol{0}, \qquad -\boldsymbol{s} \cdot \epsilon(E)\,\partial_u \boldsymbol{E} = 0. \tag{A.4}$$

$$-\boldsymbol{s} \times \partial_u \boldsymbol{E} + \mu(H)\,\partial_u \boldsymbol{H} = \boldsymbol{0}, \qquad -\boldsymbol{s} \cdot \mu(H)\,\partial_u \boldsymbol{H} = 0. \tag{A.5}$$

After elimination of either $\partial_u \boldsymbol{H}$ or $\partial_u \boldsymbol{E}$ from (A.4) and (A.5) we obtain for the slowness the condition

$$s = (\boldsymbol{s} \cdot \boldsymbol{s})^{1/2} = (\epsilon(E)\mu(H))^{1/2}. \tag{A.6}$$

By analogy with the theory of linear waves we introduce the vectorial wave impedance $\boldsymbol{Z}$ and the vectorial wave admittance $\boldsymbol{Y}$ defined by

$$\boldsymbol{Z} = \boldsymbol{s}/\epsilon(E), \qquad \boldsymbol{Y} = \boldsymbol{s}/\mu(H). \tag{A.7}$$

From (A.6) it follows that

$$Z = Z(E, H) = 1/Y(E, H) = (\mu(H)/\epsilon(E))^{1/2}, \tag{A.8}$$

while $\boldsymbol{Z} \cdot \boldsymbol{Y} = 1$. With (A.7) and (A.8) the equations (A.4) and (A.5) can be rewritten as

$$\partial_u \boldsymbol{E} = \partial_u \boldsymbol{H} \times \boldsymbol{Z}, \qquad \boldsymbol{Z} \cdot \partial_u \boldsymbol{E} = 0, \tag{A.9}$$

$$\partial_u \boldsymbol{H} = \boldsymbol{Y} \times \partial_u \boldsymbol{E}, \qquad \boldsymbol{Y} \cdot \partial_u \boldsymbol{H} = 0. \tag{A.10}$$

The power flow carried by a simple wave is described by its Poynting vector $\boldsymbol{S}(u) = \boldsymbol{E}(u) \times \boldsymbol{H}(u)$. When $w_e = w_e(u)$ and $w_m = w_m(u)$ denote the instantaneous values of the electric and the magnetic energy densities respectively, it follows that with (A.4) and (A.5) for $\partial_u \boldsymbol{S}$ can be found:

$$\partial_u \boldsymbol{S} = v(E, H)(\partial_u w_m + \partial_u w_e)\hat{\boldsymbol{s}}(E, H) - \epsilon^{-1}(E)(\boldsymbol{H} \cdot \boldsymbol{s})\, \partial_u \boldsymbol{H} - \mu^{-1}(H)(\boldsymbol{E} \cdot \boldsymbol{s})\, \partial_u \boldsymbol{E}. \tag{A.11}$$

Hence the instantaneous, local value of $\partial_u \boldsymbol{S}$ possesses a component along the slowness vector $\boldsymbol{s}$ as well as a component perpendicular to it. In (A.11), $v(E, H) = 1/s(E, H)$ denotes the local value of the velocity of the simple wave and $\hat{\boldsymbol{s}} = \hat{\boldsymbol{s}}(E, H)$ denotes the unit vector in the direction of $\boldsymbol{s}$. For the special case when $\boldsymbol{s}$ has a constant direction, we observe from (A.4) and (A.5) that $\hat{\boldsymbol{s}} \cdot \boldsymbol{E}$ and $\hat{\boldsymbol{s}} \cdot \boldsymbol{H}$ are independent of u and as a consequence

$$\boldsymbol{s} \cdot \boldsymbol{E} = 0, \qquad \boldsymbol{s} \cdot \boldsymbol{H} = 0. \tag{A.12}$$

With the aid of (A.12) we obtain after integration of (A.11)

$$\boldsymbol{S}(u) = v(E, H)(w_e + w_m)\hat{\boldsymbol{s}} - \hat{\boldsymbol{s}} \int_0^u \partial_w v(E, H)(w_e + w_m)\, \mathrm{d}w. \tag{A.13}$$

The first term on the right-hand side of (A.13) indicates that a simple wave behaves like a plane wave in a linear medium with a local velocity $v = v(E, H)$. The second term shows its difference in behaviour from a linear wave due to the fact that the wave velocity depends on E and H.

A special situation occurs when the simple waves under consideration have a constant amplitude. The so called *constant-amplitude waves* have been investigated for the first time by Carroll (1967, 1972) and by Pettini (1969). For a constant-amplitude wave we have $E = (\boldsymbol{E} \cdot \boldsymbol{E})^{1/2} = \text{constant}$, $H = (\boldsymbol{H} \cdot \boldsymbol{H})^{1/2} = \text{constant}$. It then follows that $\boldsymbol{s}$ is a constant vector. As a consequence $\boldsymbol{Z}$ and $\boldsymbol{Y}$ also become constant vectors and the equations (A.9) and (A.10) change into

$$\boldsymbol{E} = \boldsymbol{H} \times \boldsymbol{Z}, \qquad \boldsymbol{Z} \cdot \boldsymbol{E} = 0 \tag{A.14}$$

$$\boldsymbol{H} = \boldsymbol{Y} \times \boldsymbol{E}, \qquad \boldsymbol{Y} \cdot \boldsymbol{H} = 0. \tag{A.15}$$

We further find that now $\boldsymbol{E} \cdot \boldsymbol{H} = 0$. For chosen values of E and H, the medium behaves as if it were linear. With

$$w_e = \tfrac{1}{2}\epsilon(E)E^2 = w_m = \tfrac{1}{2}\mu(H)H^2, \tag{A.16}$$

we have for the Poynting vector

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = v(E, H)(w_e + w_m)\hat{\boldsymbol{s}}. \tag{A.17}$$

Equations (A.14)–(A.15) are similar to those for travelling plane waves in a linear, isotropic medium. We observe that a constant-amplitude wave allows for distortionless transmission of information through a non-dispersive, isotropic, non-linear medium.

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